Mem. S.A.It. Vol. 75, 282
© SAIt 2008

Memorie della

Gravitational waves production from stellar encounters around massive black holes

M. De Laurentis[1] and S. Capozziello[1]

Dipartimento di Scienze Fisiche, Universitá di Napoli " Federico II",
INFN Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126,
Napoli, Italy e-mail: felicia@na.infn.it

Abstract. The emission of gravitational waves from a system of massive objects interacting on elliptical, hyperbolic and parabolic orbits is studied in the quadrupole approximation. Analytical expressions are then derived for the gravitational wave luminosity, the total energy output and gravitational radiation amplitude. A crude estimate of the expected number of events towards peculiar targets (i.e. globular clusters) is also given. In particular, the rate of events per year is obtained for the dense stellar cluster at the Galactic Center.

Key words. Theory of orbits – Gravitational Waves

1. Introduction

Gravitational waves are generated by dynamical astrophysical events, and they are expected to be strong enough to be detected when compact stars such as neutron stars (NS) or black holes (BH) are involved in such events. In particular, coalescing compact binaries are considered to be the most promising sources of gravitational radiation that can be detected by the ground-based laser interferometers.Advanced optical configurations capable of reaching sensitivities slightly above and even below the so-called standard-quantum-limit for a free test-particle, have been designed for second and third generation GW detectors. A laser-interferometer space antenna (LISA) ($10^{-4} \sim 10^{-2} Hz$) might fly within the next decade. It is important in order to predict the accurate waveforms of GWs emitted by extreme mass-ratio binaries, which are among the most promising sources for LISA. To this aim, searching for criteria to classify the ways in which sources collide is of fundamental importance. A first rough criterion can be the classification of stellar encounters in *collisional* as in the globular clusters and in *collisionless* as in the galaxies J.Binney et al. (1987). A fundamental parameter is the richness and the density of the stellar system and so, obviously, we expect a large production of GWs in rich and dense systems. Systems with these features are the globular clusters and the galaxy centers. In particular, one can take into account the stars (early-type and late-type) which are around the Galactic Center, Sagittarius A^* ($SgrA^*$) which could be very interesting targets for the above mentioned ground-based and space-based detectors. In recent years, detailed information has been achieved for kinematics and dynamics of stars moving in the gravitational field of such a central object. The statistical properties of spatial and kinematical distributions

Send offprint requests to: M. De Laurentis

are of particular interest. Considering a field of resolved stars whose proper motions are accurately known, one can classify orbital motions and deduce, in principle, the rate of production of GWs according to the different types of orbits. This work is motivated by a classification of orbits in accordance with the conditions of motion, we want to calculate the GW luminosity for the different types of stellar encounters. A similar approach has been developed in S. Capozziello and M. De Laurentis (2008) but, in that case, only hyperbolic trajectories have been considered. In this report we investigate the GW emission by binary systems considering bounded (circular or elliptical) and unbounded (parabolic or hyperbolic) orbits. We expect that gravitational waves are emitted with a ”peculiar” signature related to the encounter-type: such a signature has to be a ”burst” wave-form with a maximum in correspondence of the periastron distance. The problem of *bremsstrahlung*-like gravitational wave emission has been studied in detail by Kovacs and Thorne by considering stars interacting on unbounded and bounded orbits. In this report, we face this problem discussing in detail the dynamics of such a phenomenon which could greatly improve the statistics of possible GW sources.

2. Orbits in stellar encounters

Let us take into account the Newtonian theory of orbits since stellar systems, also if at high densities and constituted by compact objects, can be usually assumed in Newtonian regime. We give here a self-contained summary of the well-known orbital types in order to achieve below a clear classification of the possible GW emissions. We refer to the text books J.Binney et al. (1987); L. Landau et al. (1973) for a detailed discussion.A mass m_1 is moving in the gravitational potential Φ generated by a second mass m_2. The vector radius and the polar angle depend on time as a consequence of the star motion, i.e. $\mathbf{r} = \mathbf{r}(t)$ and $\phi = \phi(t)$. the total energy and the angular momentum, read out

$$\frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\mathbb{L}^2}{2\mu r^2} - \frac{\gamma}{r} = E \tag{1}$$

and

$$L = r^2\frac{d\phi}{dt}, \tag{2}$$

respectively, where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the system and $\gamma = Gm_1m_2$. To solve the above differential equations we write

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{L}{\mu r^2}\frac{dr}{d\phi} = -\frac{L}{\mu}\frac{d}{d\phi}\left(\frac{1}{r}\right) \tag{3}$$

and defining, as standard, the auxiliary variable $u = 1/r$, Eq. (1) takes the form

$$u'^2 + u^2 - \frac{2\gamma\mu}{L^2}u = \frac{2\mu E}{L^2} \tag{4}$$

where $u' = du/d\phi$ and we have divided by $L^2/2\mu$. Differentiating with respect to ϕ, we get

$$u'\left(u'' + u - \frac{\gamma\mu}{L^2}\right) = 0 \tag{5}$$

hence either $u' = 0$, corresponding to the circular motion, or

$$u'' + u = \frac{\gamma\mu}{L^2} \tag{6}$$

which has the solution

$$u = \frac{\gamma\mu}{L^2} + C\cos(\phi+\alpha) \tag{7}$$

or, reverting the variable,

$$r = \left[\frac{\gamma\mu}{L^2} + C\cos(\phi+\alpha)\right]^{-1} \tag{8}$$

which is the canonical form of conic sections in polar coordinates. The constant C and α are two integration constants of the second order differential equation (6). The solution (8) must satisfy the first order differential equation (4). Substituting (8) into (4) we find, after a little algebra,

$$C^2 = \frac{2\mu E}{L^2} + \left(\frac{\gamma\mu}{L^2}\right)^2 \tag{9}$$

and we get $C^2 \geq 0$. This implies the four kinds of orbits given in Table I (see (S. Capozziello and M. De Laurentis 2008)). Circular motion correspond to

$C = 0$	$E = E_{min}$	circular orbits
$0 < \|C\| < \frac{\gamma\mu}{L^2}$	$E_{min} < E < 0$	elliptic orbits
$\|C\| = \frac{\gamma\mu}{L^2}$	$E = 0$	parabolic orbits
$\|C\| > \frac{\gamma\mu}{L^2}$	$E > 0,$	hyperbolic orbits

Table 1. Orbits in Newtonian regime classified by the approaching energy.

$$r_0 = -\frac{\gamma}{2E_{min}}. \tag{10}$$

Elliptic motion

$$r = \frac{l}{1 + \epsilon\cos\phi} \tag{11}$$

where $\epsilon = \sqrt{\frac{1-l}{a}}$ is the eccentricity of the ellipse and l is semi-latus-rectum of the ellipse. Parabolic and Hyperpolic motion correspond to

$$1 + \epsilon\cos\phi > 0 \tag{12}$$

This means $\cos\phi > -1$, i.e. $\phi \in (-\pi, \pi)$ and the trajectory is not closed any more. For $\phi \to \pm\pi$, we have $r \to \infty$. The curve, with $\epsilon = 1$, is a parabola. For $\epsilon > 1$, the allowed interval of polar angles is smaller than $\phi \in (-\pi, \pi)$, and the trajectory is a hyperbola. Such trajectories correspond to non-returning objects.

3. Gravitational wave

At this point, considering the orbit equations, we want to classify the gravitational radiation for the different stellar encounters (C. W. Misner et al. 1973). Direct signatures of gravitational radiation are its amplitude and its wave-form. In other words, the identification of a GW signal is strictly related to the accurate selection of the shape of wave-forms by interferometers or any possible detection tool. Such an achievement could give information on the nature of the GW source, on the propagating medium, and , in principle, on the gravitational theory producing such a radiation. It is well known that the amplitude of GWs can be evaluated by

$$h^{jk}(t, R) = \frac{2G}{Rc^4}\ddot{Q}^{jk} , \tag{13}$$

R being the distance between the source and the observer and $\{j, k\} = 1, 2$, where Q_{ij} is the quadrupole mass tensor

$$Q_{ij} = \sum_a m_a(3x_a^i x_a^j - \delta_{ij} r_a^2) , \tag{14}$$

. Here G being the Newton constant, r_a the modulus of the vector radius of the $a - th$ particle and the sum running over all masses m_a in the system.We now derive the GW amplitude in relation to the orbital shape of the binary systems.As an example, the amplitude of gravitational wave is sketched in Fig. 1 for a stellar encounter close to the Galactic Center. The adopted initial parameters are typical of a close impact and are assumed to be $b = 1$ AU and $v_0 = 200$ Kms^{-1}, respectively. Here, we have fixed $M_1 = M_2 = 1.4M_\odot$. The impact parameter is defined as $L = bv$ where L is the angular momentum and v the incoming velocity. We have chosen a typical velocity of a star in the galaxy and we are considering, essentially, compact objects with masses comparable to the Chandrasekhar limit ($\sim 1.4M_\odot$). This choice is motivated by the fact that ground-based experiments like VIRGO or LIGO expect to detect typical GW emissions from the dynamics of these objects or from binary systems composed by them.

4. Rate and event number estimations

An important remark is due at this point. A galaxy is a self-gravitating collisionless system where stellar impacts are very rare (J.Binney et al. 1987). From the GW emission point of view, close orbital encounters, collisions and tidal interactions should be dealt on average if we want to investigate the gravitational radiation in a dense stellar system as we are going to do now.

Let us give now an estimate of the stellar encounter rate producing GWs in some interesting astrophysical conditions like a typical

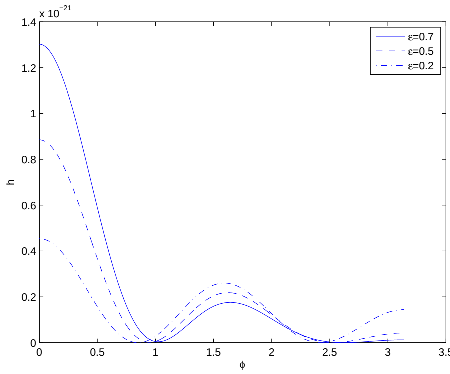

Fig. 1. The gravitational wave-forms from elliptical orbits shown as function of the polar angle ϕ. We have fixed $M_1 = M_2 = 1.4M_\odot$. M_2 is considered at rest while M_1 is moving with initial velocity $v_0 = 200$ Kms^{-1} and an impact parameter $b = 1$ AU. The distance of the GW source is assumed to be $R = 8$ kpc and the eccentricity is $\epsilon = 0.2, 0.5, 0.7$.

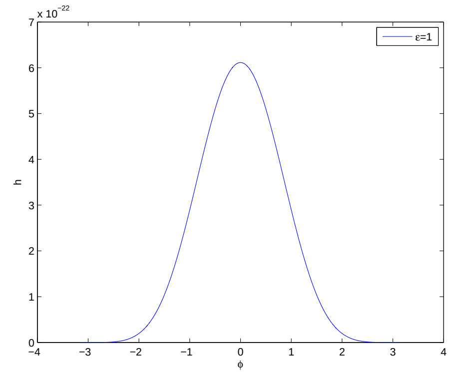

Fig. 2. The gravitational wave-forms for a parabolic encounter as a function of the polar angle ϕ. As above, $M_1 = M_2 = 1.4M_\odot$ and M_2 is considered at rest. M_1 is moving with initial velocity $v_0 = 200$ Kms^{-1} with an impact parameter $b = 1$ AU. The distance of the GW source is assumed at $R = 8$ kpc. The eccentricity is $\epsilon = 1$.

globular cluster or towards the Galactic Center after we have discussed above the features distinguishing the various types of stellar encounters. Up to now, we have approximated stars as point masses. However, in dense regions of stellar systems, a star can pass so close to another that they raise tidal forces which dissipate their relative orbital kinetic energy. In some cases, the loss of energy can be so large that stars form binary or multiple systems; in other cases, the stars collide and coalesce into a single star; finally stars can exchange gravitational interaction in non-returning encounters. To investigate and parameterize all these effects, we have to compute the collision time t_{coll}, where $1/t_{coll}$ is the collision rate, that is, the average number of physical collisions that a given star suffers per unit time. For the sake of simplicity, we restrict to stellar clusters in which all stars have the same mass m. Let us consider an encounter with initial relative velocity $\mathbf{v}_0$ and impact parameter b. The angular momentum per unit mass of the reduced particle is $L = bv_0$. At the distance of closest approach, which we denote by r_{coll}, the radial velocity must be zero, and hence the angular momentum is $L = r_{coll}v_{max}$, where v_{max} is the relative speed at r_{coll}. From the energy equation (1), we have

$$b^2 = r_{coll}^2 + \frac{4Gmr_{coll}}{v_0^2} \,. \tag{15}$$

If we set r_{coll} equal to the sum of the radii of two stars, then a collision will occur if and only if the impact parameter is less than the value of b, as determined by Eq.(15).

Let $f(\mathbf{v}_a)d^3\mathbf{v}_a$ be the number of stars per unit volume with velocities in the range $\mathbf{v}_a + d^3\mathbf{v}_a$. The number of encounters per unit time with impact parameter less than b which are suffered by a given star is just $f(\mathbf{v}_a)d^3\mathbf{v}_a$ times the volume of the annulus with radius b and length v_0, that is,

$$\int f(\mathbf{v}_a)\pi b^2 v_0 d^3\mathbf{v}_a \tag{16}$$

where $v_0 = |\mathbf{v} - \mathbf{v}_a|$ and $\mathbf{v}$ is the velocity of the considered star. The quantity in Eq.(16) is equal to $1/t_{coll}$ for a star with velocity $\mathbf{v}$: to obtain the mean value of $1/t_{coll}$, we average over $\mathbf{v}$ by multiplying (16) by $f(\mathbf{v})/\nu$, where

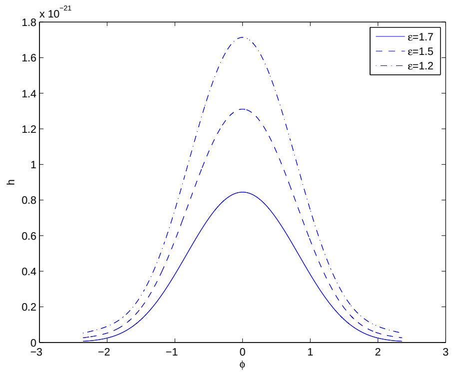

Fig. 3. The gravitational wave-forms for hyperbolic encounters as function of the polar angle ϕ. As above, we have fixed $M_1 = M_2 = 1.4M_\odot$. M_2 is considered at rest while M_1 is moving with initial velocity $v_0 = 200$ Kms^{-1} and an impact parameter $b = 1$ AU. The distance of the source is assumed at $R = 8$ kpc. The eccentricity is assumed with the values $\epsilon = 1.2, 1.5, 1.7$.

$\nu = \int f(\mathbf{v}) d^3\mathbf{v}$ is the number density of stars and the integration is over $d^3\mathbf{v}$. Thus

$$\frac{1}{t_{coll}} = \frac{\nu}{8\pi^2\sigma^6}\int e^{-(v^2+v_a^2)/2\sigma^2} \left(r_{coll}\,|\mathbf{v}-\mathbf{v}_a| + \frac{4Gmr_{coll}}{|\mathbf{v}-\mathbf{v}_a|}\right) d^3\mathbf{v} d^3\mathbf{v}_a \qquad (17)$$

We now replace the variable $\mathbf{v}_a$ by $\mathbf{V} = \mathbf{v} - \mathbf{v}_a$. The argument of the exponential is then $-\left[\left(\mathbf{v} - \frac{1}{2}\mathbf{V}\right)^2 + \frac{1}{4}V^2\right]/\sigma^2$, and if we replace the variable $\mathbf{v}$ by $\mathbf{v}_{cm} = \mathbf{v} - \dfrac{1}{2}\mathbf{V}$ (the center of mass velocity), then we have

$$\frac{1}{t_{coll}} = \frac{\nu}{8\pi^2\sigma^6} \int e^{-(v_{cm}^2+V^2)/2\sigma^2}\left(r_{coll}V + \frac{4Gmr_{coll}}{V}\right) dV \,. \qquad (18)$$

The integral over $\mathbf{v}_{cm}$ is given by

$$\int e^{-v_{cm}^2/\sigma^2} d^3\mathbf{v}_{cm} = \pi^{3/2}\sigma^3 \,. \qquad (19)$$

Thus

$$\frac{1}{t_{coll}} = \frac{\pi^{1/2}\nu}{2\sigma^3}\int_\infty^0 e^{-V^2/4\sigma^2} \left(r_{coll}^2 V^3 + 4GmVr_{coll}\right) dV \qquad (20)$$

The integrals can be easily calculated and then we find

$$\frac{1}{t_{coll}} = 4\sqrt{\pi}\nu\sigma r_{coll}^2 + \frac{4\sqrt{\pi}\nu Gmr_{coll}}{\sigma}\,. \qquad (21)$$

The first term of this result can be derived from the kinetic theory. The rate of interaction is $\nu\Sigma\langle V\rangle$, where Σ is the cross-section and $\langle V\rangle$ is the mean relative speed. Substituting $\Sigma = \pi r_{coll}^2$ and $\langle V\rangle = 4\sigma/\sqrt{\pi}$ (which is appropriate for a Maxwellian distribution whit dispersion σ) we recover the first term of (21). The second term represents the enhancement in the collision rate by gravitational focusing, that is, the deflection of trajectories by the gravitational attraction of the two stars.

If r_* is the stellar radius, we may set $r_{coll} = 2r_*$. It is convenient to introduce the escape speed from stellar surface, $v_* = \sqrt{\dfrac{2Gm}{r_*}}$, and to rewrite Eq.(21) as

$$\Gamma = \frac{1}{t_{coll}} = 16\sqrt{\pi}\nu\sigma r_*^2\left(1 + \frac{v_*^2}{4\sigma^2}\right) = 16\sqrt{\pi}\nu\sigma r_*^2(1+\Theta), \qquad (22)$$

where

$$\Theta = \frac{v_*^2}{4\sigma^2} = \frac{Gm}{2\sigma^2 r_*} \qquad (23)$$

is the Safronov number (J.Binney et al. 1987). In evaluating the rate, we are considering only those encounters producing gravitational waves, for example, in the LISA range, i.e. between 10^{-4} and 10^{-2} Hz . Numerically, we have

$$\Gamma \simeq 5.5\times 10^{-10}\left(\frac{\nu}{10\mathrm{kms}^{-1}}\right)\left(\frac{\sigma}{UA^2}\right)\left(\frac{10\mathrm{pc}}{R}\right)^3 \mathrm{yrs}^{-1} \qquad \Theta << 1 \qquad (24)$$

$$\Gamma \simeq 5.5 \times 10^{-10} \left(\frac{M}{10^5 \mathrm{M}_\odot}\right)^2 \left(\frac{v}{10\mathrm{kms}^{-1}}\right) \times \left(\frac{\sigma}{UA^2}\right) \left(\frac{10\mathrm{pc}}{R}\right)^3 \mathrm{yrs}^{-1} \qquad \Theta >> 1 \qquad (25)$$

If $\Theta >> 1$, the energy dissipated exceeds the relative kinetic energy of the colliding stars, and the stars coalesce into a single star. This new star may, in turn, collide and merge with other stars, thereby becoming very massive. As its mass increases, the collision time is shorten and then there may be runaway coalescence leading to the formation of a few supermassive objects per clusters. If $\Theta << 1$, much of the mass in the colliding stars may be liberated and forming new stars or a single supermassive objects.

Note that when we have the effects of quasi-collisions in an encounter of two stars in which the minimum separation is several stellar radii, violent tides will raise on the surface of each star. The energy that excites the tides comes from the relative kinetic energy of the stars. This effect is important for $\Theta >> 1$ since the loss of small amount of kinetic energy may leave the two stars with negative total energy, that is, as a bounded binary system. Successive peri-center passages will dissipates more energy by GW radiation, until the binary orbit is nearly circular with a negligible or null GW radiation emission.

Let us apply these considerations to the Galactic Center which can be modelled as a system of several compact stellar clusters, some of them similar to very compact globular clusters with high emission in X-rays.

For a typical **compact stellar cluster** around the Galactic Center, the expected event rate is of the order of 2×10^{-9} yrs^{-1} which may be increased at least by a factor $\simeq 100$ if one considers the number of globular clusters in the whole Galaxy eventually passing nearby the Galactic Center. If the compact stellar cluster at the Galactic Center is taken into account and assuming the total mass $M \simeq 3 \times 10^6$ $\mathrm{M}_\odot$, the velocity dispersion $\sigma \simeq 150$ km s^{-1} and the radius of the object $R \simeq 10$ pc (where $\Theta = 4.3$), one expects to have $\simeq 10^{-5}$ open orbit encounters per year. On the other hand, if a cluster with total mass $M \simeq 10^6$ $\mathrm{M}_\odot$, $\sigma \simeq 150$ km s^{-1} and $R \simeq 0.1$ pc is considered, an event rate number of the order of unity per year is obtained. These values could be realistically achieved by data coming from the forthcoming space interferometer LISA. As a secondary effect, the above wave-forms could constitute the ”signature” to classify the different stellar encounters thanks to the differences of the shapes (see the above figures).

5. Concluding Remarks

We have analyzed the gravitational wave emission coming from stellar encounters in Newtonian regime and in quadrupole approximation. In particular, we have taken into account the expected luminosity and the strain amplitude of gravitational radiation produced in tight impacts where two massive objects of $1.4M_\odot$ closely interact at an impact distance of $1AU$. Due to the high probability of such encounters inside rich stellar fields (e.g. globular clusters, bulges of galaxies and so on), the presented approach could highly contribute to enlarge the classes of gravitational wave sources (in particular, of dynamical phenomena capable of producing gravitational waves). In particular, a detailed theory of stellar orbits could improve the statistic of possible sources.

References

A.M. Ghez, B.L. Klein, M. Morris and E.E.Becklin, Ap. J. 509, 678 (1998).

J.Binney and S.Tremaine, *Galactic Dynamics*, Princeton University Press, Princeton, New Jersey (1987).

L. Landau et al., *Mechanics*, Pergamon Press, New York (1973).

S. Capozziello S. and M. De Laurentis, Astropart.Phys., Vol **30**, 105-112 (2008).

C. W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman, New. York (1973).